Use a CAS double-integral evaluator to find the integrals in Exercises 97–102. Then reverse the order of integration and evaluate, again with a CAS.

97. $\int_0^1 \int_{2y}^4 e^{x^2}\, dx\, dy$

98. $\int_0^3 \int_{x^2}^9 x \cos(y^2)\, dy\, dx$

99. $\int_0^2 \int_{y^3}^{4\sqrt{2y}} (x^2y - xy^2)\, dx\, dy$

100. $\int_0^2 \int_0^{4-y^2} e^{xy}\, dx\, dy$

101. $\int_1^2 \int_0^{x^2} \frac{1}{x+y}\, dy\, dx$

102. $\int_1^2 \int_{y^3}^8 \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy$

15.3 Area by Double Integration

In this section we show how to use double integrals to calculate the areas of bounded regions in the plane, and to find the average value of a function of two variables.

Areas of Bounded Regions in the Plane

If we take $f(x, y) = 1$ in the definition of the double integral over a region R in the preceding section, the Riemann sums reduce to

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k = \sum_{k=1}^{n} \Delta A_k. \tag{1}$$

This is simply the sum of the areas of the small rectangles in the partition of R, and it approximates what we would like to call the area of R. As the norm of a partition of R approaches zero, the height and width of all rectangles in the partition approach zero, and the coverage of R becomes increasingly complete (Figure 15.8). We define the area of R to be the limit

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta A_k = \iint_R dA. \tag{2}$$

DEFINITION The **area** of a closed, bounded plane region R is

$$A = \iint_R dA.$$

As with the other definitions in this chapter, the definition here applies to a greater variety of regions than does the earlier single-variable definition of area, but it agrees with the earlier definition on regions to which they both apply. To evaluate the integral in the definition of area, we integrate the constant function $f(x, y) = 1$ over R.

EXAMPLE 1 Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution We sketch the region (Figure 15.19), noting where the two curves intersect at the origin and $(1, 1)$, and calculate the area as

$$A = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 \Big[y \Big]_{y=x^2}^{y=x} dx = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$

Notice that the single-variable integral $\int_0^1 (x - x^2)\, dx$, obtained from evaluating the inside iterated integral, is the integral for the area between these two curves using the method of Section 5.6. ■

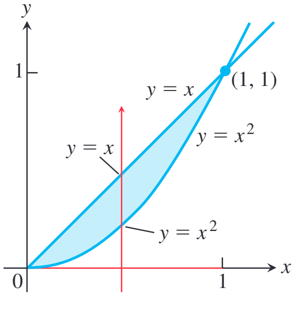

FIGURE 15.19 The region in Example 1.

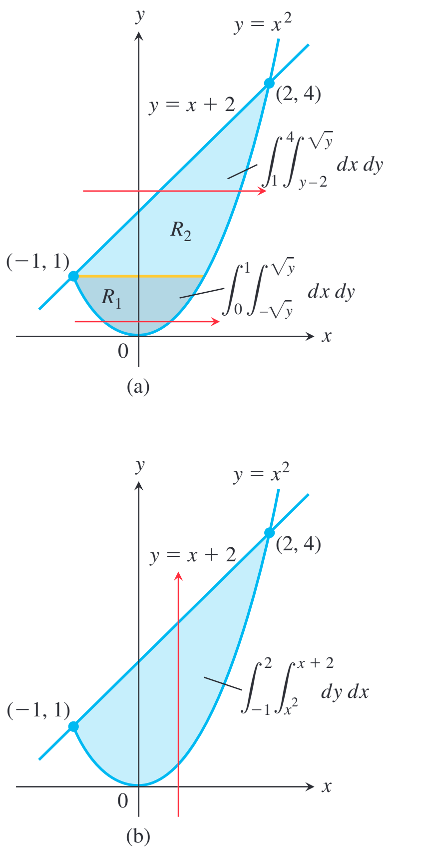

FIGURE 15.20 Calculating this area takes (a) two double integrals if the first integration is with respect to x, but (b) only one if the first integration is with respect to y (Example 2).

EXAMPLE 2 Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Solution If we divide R into the regions R_1 and R_2 shown in Figure 15.20a, we may calculate the area as

$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\,dy.$$

On the other hand, reversing the order of integration (Figure 15.20b) gives

$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy\,dx.$$

This second result, which requires only one integral, is simpler to evaluate, giving

$$A = \int_{-1}^{2} \left[y\right]_{y=x^2}^{y=x+2} dx = \int_{-1}^{2} (x + 2 - x^2)\,dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}.$$

EXAMPLE 3 Find the area of the playing field described by R: $-2 \le x \le 2,\ -1 - \sqrt{4 - x^2} \le y \le 1 + \sqrt{4 - x^2}$, using

(a) Fubini's Theorem **(b)** simple geometry.

Solution The region R is shown in Figure 15.21a.

(a) From the symmetries observed in the figure, we see that the area of R is 4 times its area in the first quadrant. As shown in Figure 15.21b, a vertical line at x enters this part of the region at $y = 0$ and exits at $y = 1 + \sqrt{4 - x^2}$. Therefore, using Fubini's Theorem, we have

$$\begin{aligned} A = \iint_R dA &= 4\int_0^2 \int_0^{1+\sqrt{4-x^2}} dy\,dx \\ &= 4\int_0^2 \left(1 + \sqrt{4 - x^2}\right) dx \\ &= 4\left[x + \frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 \qquad \text{Integral Table Formula 45} \\ &= 4\left(2 + 0 + 2\cdot\frac{\pi}{2} - 0\right) = 8 + 4\pi. \end{aligned}$$

(b) The region R consists of a rectangle mounted on two sides by half disks of radius 2. The area can be computed by summing the area of the 4×2 rectangle and the area of a circle of radius 2, so

$$A = 8 + \pi 2^2 = 8 + 4\pi.$$

Average Value

The average value of an integrable function of one variable on a closed interval is the integral of the function over the interval divided by the length of the interval. For an integrable function of two variables defined on a bounded region in the plane, the average value is the integral over the region divided by the area of the region. This can be visualized by thinking of the region as being the base of a tank with vertical walls around the boundary of the region, and imagining that the tank is filled with water that is sloshing around. The value

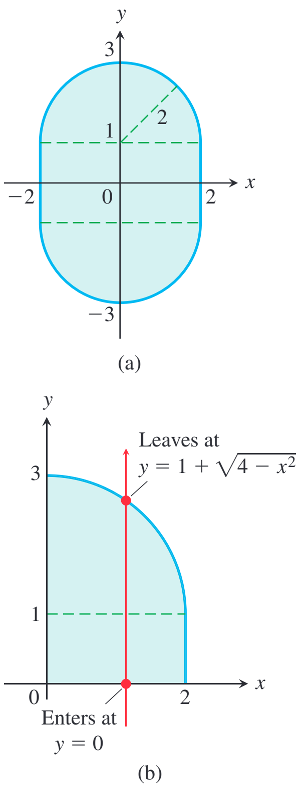

FIGURE 15.21 (a) The playing field described by the region R in Example 3. (b) First quadrant of the playing field.

$f(x, y)$ is then the height of the water that is directly above the point (x, y). The average height of the water in the tank can be found by letting the water settle down to a constant height. This height is equal to the volume of water in the tank divided by the area of R. We therefore define the average value of an integrable function f over a region R as follows:

$$\textbf{Average value } \text{of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f\, dA. \tag{3}$$

If f is the temperature of a thin plate covering R, then the double integral of f over R divided by the area of R is the plate's average temperature. If $f(x, y)$ is the distance from the point (x, y) to a fixed point P, then the average value of f over R is the average distance of points in R from P.

EXAMPLE 4 Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R: 0 \le x \le \pi,\ 0 \le y \le 1$.

Solution The value of the integral of f over R is

$$\int_0^{\pi}\int_0^1 x \cos xy\, dy\, dx = \int_0^{\pi} \Big[\sin xy\Big]_{y=0}^{y=1} dx \qquad \int x \cos xy\, dy = \sin xy + C$$

$$= \int_0^{\pi} (\sin x - 0)\, dx = -\cos x\Big]_0^{\pi} = 1 + 1 = 2.$$

The area of R is π. The average value of f over R is $2/\pi$.

EXERCISES 15.3

Area by Double Integrals

In Exercises 1–12, sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral.

1. The coordinate axes and the line $x + y = 2$
2. The lines $x = 0$, $y = 2x$, and $y = 4$
3. The parabola $x = -y^2$ and the line $y = x + 2$
4. The parabola $x = y - y^2$ and the line $y = -x$
5. The curve $y = e^x$ and the lines $y = 0$, $x = 0$, and $x = \ln 2$
6. The curves $y = \ln x$ and $y = 2 \ln x$ and the line $x = e$, in the first quadrant
7. The parabolas $x = y^2$ and $x = 2y - y^2$
8. The parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$
9. The lines $y = x$, $y = x/3$, and $y = 2$
10. The lines $y = 1 - x$ and $y = 2$ and the curve $y = e^x$
11. The lines $y = 2x$, $y = x/2$, and $y = 3 - x$
12. The lines $y = x - 2$ and $y = -x$ and the curve $y = \sqrt{x}$

Identifying the Region of Integration

The integrals and sums of integrals in Exercises 13–18 give the areas of regions in the xy-plane. Sketch each region, label each bounding curve with its equation, and give the coordinates of the points where the curves intersect. Then find the area of the region.

13. $\displaystyle\int_0^6 \int_{y^2/3}^{2y} dx\, dy$
14. $\displaystyle\int_0^3 \int_{-x}^{x(2-x)} dy\, dx$
15. $\displaystyle\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\, dx$
16. $\displaystyle\int_{-1}^2 \int_{y^2}^{y+2} dx\, dy$
17. $\displaystyle\int_{-1}^0 \int_{-2x}^{1-x} dy\, dx + \int_0^2 \int_{-x/2}^{1-x} dy\, dx$
18. $\displaystyle\int_0^2 \int_{x^2-4}^0 dy\, dx + \int_0^4 \int_0^{\sqrt{x}} dy\, dx$

Finding Average Values

19. Find the average value of $f(x, y) = \sin(x + y)$ over
    - **a.** the rectangle $0 \le x \le \pi$, $0 \le y \le \pi$.
    - **b.** the rectangle $0 \le x \le \pi$, $0 \le y \le \pi/2$.
20. Which do you think will be larger, the average value of $f(x, y) = xy$ over the square $0 \le x \le 1$, $0 \le y \le 1$, or the average value of f over the quarter circle $x^2 + y^2 \le 1$ in the first quadrant? Calculate them to find out.
21. Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2$, $0 \le y \le 2$.

22. Find the average value of $f(x, y) = 1/(xy)$ over the square $\ln 2 \le x \le 2\ln 2,\ \ln 2 \le y \le 2\ln 2$.

Theory and Examples

23. Geometric area Find the area of the region

$$R:\ 0 \le x \le 2,\ 2 - x \le y \le \sqrt{4 - x^2},$$

using **(a)** Fubini's Theorem, **(b)** simple geometry.

24. Geometric area Find the area of the circular washer with outer radius 2 and inner radius 1, using **(a)** Fubini's Theorem, **(b)** simple geometry.

25. Bacterium population If $f(x, y) = (10{,}000e^y)/(1 + |x|/2)$ represents the "population density" of a certain bacterium on the xy-plane, where x and y are measured in centimeters, find the total population of bacteria within the rectangle $-5 \le x \le 5$ and $-2 \le y \le 0$.

26. Regional population If $f(x, y) = 100(y + 1)$ represents the population density of a planar region on Earth, where x and y are measured in miles, find the number of people in the region bounded by the curves $x = y^2$ and $x = 2y - y^2$.

27. Average temperature in Texas According to the *Texas Almanac*, Texas has 254 counties and a National Weather Service station in each county. Assume that at time t_0, each of the 254 weather stations recorded the local temperature. Find a formula that would give a reasonable approximation of the average temperature in Texas at time t_0. Your answer should involve information that you would expect to be readily available in the *Texas Almanac*.

28. If $y = f(x)$ is a nonnegative continuous function over the closed interval $a \le x \le b$, show that the double integral definition of area for the closed plane region bounded by the graph of f, the vertical lines $x = a$ and $x = b$, and the x-axis agrees with the definition for area beneath the curve in Section 5.3.

29. Suppose $f(x, y)$ is continuous over a region R in the plane and that the area $A(R)$ of the region is defined. If there are constants m and M such that $m \le f(x, y) \le M$ for all $(x, y) \in R$, prove that

$$mA(R) \le \iint_R f(x, y)\, dA \le MA(R).$$

30. Suppose $f(x, y)$ is continuous and nonnegative over a region R in the plane with a defined area $A(R)$. If $\iint_R f(x, y)\, dA = 0$, prove that $f(x, y) = 0$ at every point $(x, y) \in R$.

15.4 Double Integrals in Polar Form

Double integrals are sometimes easier to evaluate if we change to polar coordinates. This section shows how to accomplish the change and how to evaluate double integrals over regions whose boundaries are given by polar equations.

Integrals in Polar Coordinates

When we defined the double integral of a function over a region R in the xy-plane, we began by cutting R into rectangles whose sides were parallel to the coordinate axes. These were the natural shapes to use because their sides have either constant x-values or constant y-values. In polar coordinates, the natural shape is a "polar rectangle" whose sides have constant r- and θ-values. To avoid ambiguities when describing the region of integration with polar coordinates, we use polar coordinate points (r, θ) where $r \ge 0$.

Suppose that a function $f(r, \theta)$ is defined over a region R that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$. Suppose also that $0 \le g_1(\theta) \le g_2(\theta) \le a$ for every value of θ between α and β. Then R lies in a fan-shaped region Q defined by the inequalities $0 \le r \le a$ and $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$. See Figure 15.22.

We cover Q by a grid of circular arcs and rays. The arcs are cut from circles centered at the origin, with radii $\Delta r, 2\Delta r, \dots, m\Delta r$, where $\Delta r = a/m$. The rays are given by

$$\theta = \alpha, \qquad \theta = \alpha + \Delta\theta, \qquad \theta = \alpha + 2\Delta\theta, \qquad \dots, \qquad \theta = \alpha + m'\Delta\theta = \beta,$$

where $\Delta\theta = (\beta - \alpha)/m'$. The arcs and rays partition Q into small patches called "polar rectangles."

We number the polar rectangles that lie inside R (the order does not matter), calling their areas $\Delta A_1, \Delta A_2, \dots, \Delta A_n$. We let (r_k, θ_k) be any point in the polar rectangle whose area is ΔA_k. We then form the sum

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\, \Delta A_k.$$